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Spherical cap bubbles with laminar wakes

By JEAN-YVES PARLANGE
Department of Engineering and Applied Science,
Yale University, New Haven, Connecticut

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The steady rise of a given amount of gas in an infinite liquid under the action of buoyancy forces is examined. The shape of the bubble is that of a spherical cap. Behind the bubble there is a region of closed streamlines where the flow is assumed to be steady and rotational. The predicted velocity and geometry of the cap agree well with observation for a limited range of values of the Reynolds number.

1. Introduction

The motion of a gas bubble, rising steadily in an infinite liquid under the action of gravity, is primarily affected by the kinematic viscosity of the liquid, the surface tension of the gas–liquid interface and the presence of impurities in the liquid.

If the liquid contains enough impurities, adsorption on the interface may create a shield. The liquid moves around the gas bubble as if the latter were made of solid material. In our study we assume that the liquid contains no surfactants in appreciable quantity and the interface is free to move.

When the size of the bubble is sufficiently small, surface tension forces predominate and the bubble has a spherical shape. As the size of the bubble increases, dynamic forces due to the flow of liquid begin to affect the shape of the bubble which changes from a sphere to an oblate ellipsoid. By further increasing the amount of gas a point is reached when surface tension is entirely negligible. In this case the shape of the bubble is that of a spherical cap with a large wake behind it. Two different situations may arise depending on the value of the Reynolds number (based on a characteristic dimension of the wake). If this number is sufficiently large, the bubble has a wobbling motion, more or less pronounced, and the flow in the wake is turbulent (Davies & Taylor 1950). If the Reynolds number is not too high (e.g. less than 10^3) the motion remains steady and the flow is laminar everywhere. We shall concern ourselves only with this last case, assuming for simplicity that the Reynolds number is much larger than one (but not so high that the flow ceases to be laminar). Let us first give a qualitative description of such a flow. The front part of the bubble is nearly spherical and the bottom is practically flat. Compare Collins' (1966) photograph where the bottom is rather irregular, due to turbulence, with Haberman & Morton's (1956) smooth picture using mineral oil. More precisely (Davenport, Richardson & Bradshaw

1967) for a laminar flow the bottom is slightly concave (see figure 1). Surface tension again has an influence on the curvature of the bubble where the front and the bottom parts join; this is of no importance for the rest of the flow. Behind the cap there is a region of closed streamlines where the flow is steady and rotational. The flow being axisymmetric, we call (L) the closed streamline, that separates the closed flow from the outside, in any plane passing through the axis of symmetry. Outside (L) the flow is irrotational, except for a thin boundary layer near (L), since we assume a very large Reynolds number.

It may be seen that for a steady flow of this kind the volume of the bubble is a diminishing fraction of the volume of the closed wake region as the Reynolds number of the bubble motion increases.† We are primarily interested in the asymptotic (zeroth-order) solution when the Reynolds number becomes very large. In that case the volume of the bubble is negligible compared with that of the closed region, and can be ignored (to the zeroth order) when determining the liquid streamlines inside and outside the wake.

Rippin & Davidson (1967) studied the same problem for an infinite and stagnant wake. In their words: "The free streamline theory has the advantage of a complete mathematical formulation, albeit based on unrealistic assumptions about the flow in the wake." It is interesting that their theory agrees fairly well with experiments in the case of a *turbulent* wake. On the other hand the present theory looks for a more realistic wake but is necessarily restricted to *laminar* flows.

2. Asymptotic solution

We consider only the motion of the liquid, ignoring the small volume occupied by the gas. The flow is axisymmetric and the system of co-ordinates is indicated on figure 1. We call U the steady upward velocity of the bubble. Let ρ be the density, p the pressure, μ the viscosity, ν $(=\mu/\rho)$ the kinematic viscosity. Instead of a bubble moving in a medium at rest, we rather consider a motionless bubble the liquid having the uniform velocity U at infinity. If $2a$ is a characteristic dimension of the wake, the Reynolds number of the flow is

$$R_e = 2aU/\nu. \tag{1}$$

Because $R_e \gg 1$, outside (L), the flow is irrotational except across a layer whose thickness goes to zero as $R_e \to \infty$. Since the boundary-layer thickness is very small, to the zeroth order we commit only a negligible error in considering (L) to be a

† The relative magnitude of these two volumes can be estimated precisely. The author is grateful to Prof. G. K. Batchelor for suggesting the following derivation:

Call a a characteristic dimension of the closed region, V the volume of the bubble, U its velocity, μ the viscosity and R_e the Reynolds number based on a. At high Reynolds number and in the absence of a rigid boundary, the total viscous dissipation is proportional to $\mu a^3(U/a)^2$. Davies & Taylor (1950) have shown that U^2 is proportional to the radius of curvature R of the upper surface of the bubble. Assuming that R and a are of the same order, then the dissipation is proportional to Ua^3/R_e. The dissipation is equal to the rate of change in potential energy, itself proportional to VU. Consequently, (volume of bubble)/(volume of closed region), which is proportional to V/a^3, is of order R_e^{-1}.

streamline of the outside irrotational flow. To the same order an application of Batchelor's (1956) theorem indicates that

$$\xi/r = A \tag{2}$$

inside (L), where ξ is the vorticity and A is a constant. Furthermore, it is quite clear that the pressures due to the flow in the wake and the irrotational flow must be equal (to the zeroth order) on (L). Since Bernoulli's equation holds to the same order, the tangential velocities must match on (L). Let us denote by ψ the stream function. At infinity the flow is uniform (velocity U), hence ψ must behave like,

$$\psi \simeq \tfrac{1}{2}Ur^2 \text{ at infinity.} \tag{3}$$

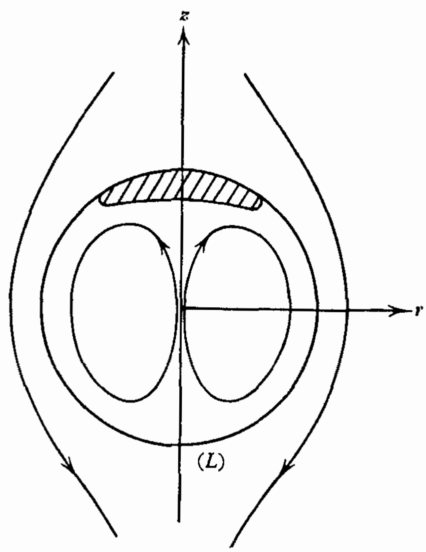

FIGURE 1. Sketch of streamlines around a spherical cap bubble.

Since ψ is defined up to an arbitrary constant we can always write

$$\psi = 0 \quad \text{on} \quad (L). \tag{4}$$

Also from the matching of tangential velocities

$$\partial\psi/\partial r \quad \text{is continuous at} \quad (L). \tag{5}$$

Equations (3), (4), (5) represent the boundary conditions associated with (2) inside (L) and $\xi = 0$ outside (L) or from the definition of vorticity

$$\frac{\xi}{r} \equiv \frac{1}{r^2}\left[\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right] = \begin{cases} A & \text{inside} \quad (L), \\ 0 & \text{outside} \quad (L). \end{cases} \tag{6}$$

Notice that viscosity has disappeared entirely from the set of equations (3) to (6) which we have to solve. Of course, viscosity plays an essential role in our problem as it is responsible for the diffusion of vorticity in the wake (hence Batchelor's theorem and equation (6)).

Suppose we take (L) to be an arbitrary simple curve, then it is well known that (3)–(6) have a unique solution, for a given U and A, if we only replace (4) by the condition that ψ be continuous at (L) (rather than constant). Insisting then that ψ must be zero on (L) gives a condition for the determination of (L). It is plausible to suppose that this condition determines (L) uniquely, but we have not been able to prove this. However, it is easy to verify that a solution is

$$\psi = \tfrac{1}{10}Ar^2(r^2+z^2-a^2) \quad \text{inside } (L), \tag{7}$$

$$\psi = \tfrac{1}{2}Ur^2[1-a^3/(r^2+z^2)^{\frac{3}{2}}] \quad \text{outside } (L), \tag{8}$$

$$A = \tfrac{15}{2}(U/a^2); \tag{9}$$

(L) is a circle of radius a, given by (9) in terms of A and U. The inside solution is the well known Hill's vortex.

The flow field is entirely determined but for A and U which are as yet unknown. The gas bubble lies at the top stagnation point within (L) (see figure 1). Inside the bubble the pressure is uniform since we can neglect the dynamic pressure induced by the circulation of the gas whose density is negligible. Hence the pressure exerted by the outside liquid on the gas must be constant. From this condition we deduce at once (see Davies & Taylor 1950)

$$\tfrac{9}{4}U^2 = ga. \tag{10}$$

An additional equation is obtained from the conservation of energy. The viscous dissipation, Φ, is easily computed (see Harper & Moore 1968).

$$\Phi = 30\pi\mu U^2 a. \tag{11}$$

From conservation of energy we must then have $\Phi = DU$, where D is the drag

$$D = \rho g V, \tag{12}$$

where V is the volume of the gas bubble. Equations (11) and (12) can be combined and give at once

$$U = \frac{1}{3}\left(\frac{4}{10\pi}\right)^{\frac{1}{3}} g^{\frac{2}{3}}\nu^{-\frac{1}{3}}V^{\frac{1}{3}}. \tag{13}$$

Figure 2 indicates the experimental results for mineral oil and the theoretical curve obtained from (13). Experimentally it was found that around $V^{\frac{1}{3}} \simeq 1{\cdot}5$ cm the bubble changes shape from an oblate ellipsoid to a spherical cap (Haberman & Morton 1956). At that point $R_e = 165$, which is sufficiently high for our theory to be adequate. Figure 3 indicates similar results for an aqueous solution with $6{\cdot}1\,\%$ of polyvinyl alcohol. In this experiment bubbles were produced in a 15 cm diameter tube; when $V^{\frac{1}{3}} \simeq 2$ cm interactions with the wall slow down the bubble significantly (see Davenport, Richardson & Bradshaw 1967).

The geometrical characteristics of the spherical cap are easily determined in terms of V and a (assuming the bottom to be flat). For instance, calling θ half the angle subtended by the bubble at the centre of the sphere we have,

$$\cos^3\theta - 3\cos\theta + 2 = 3V/\pi a^3. \tag{14}$$

The height, h and basal radius, b are given at once in terms of θ and a by,

$$h = a(1-\cos\theta), \tag{15}$$

$$b = a\sin\theta. \tag{16}$$

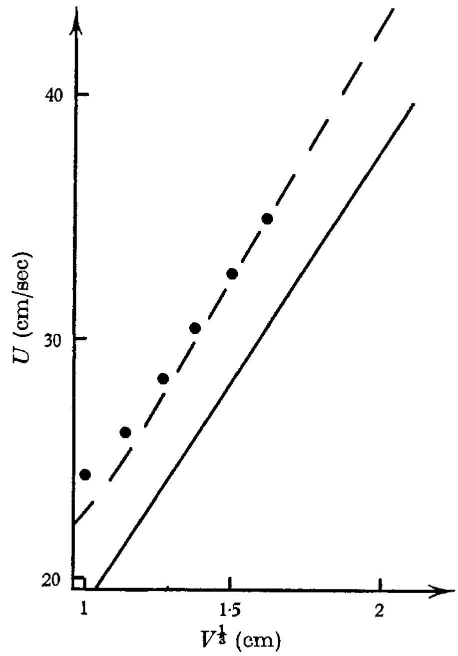

FIGURE 2. Comparison of velocity of rise from (13) (lower curve) and (20) (dashed line) with experimental results (dotted line) for mineral oil (Haberman & Morton 1956). The spherical cap was observed for $V^{\frac{1}{3}} > 1{\cdot}5$ cm.

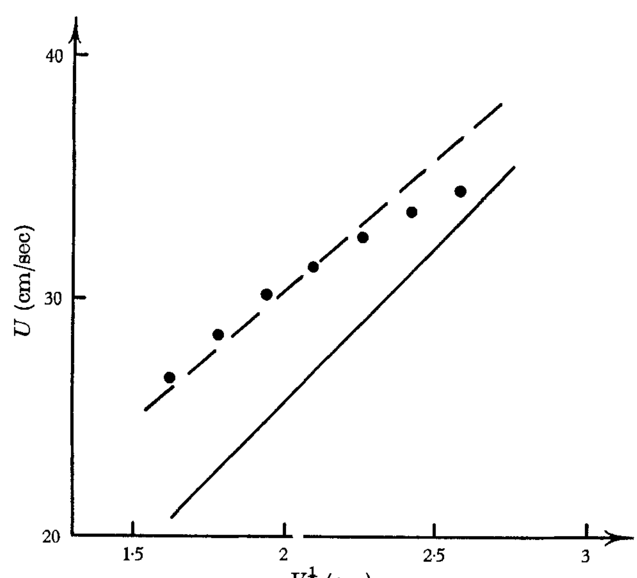

FIGURE 3. Comparison of velocity of rise from (13) (lower curve) and (20) (dashed line) with experimental results (dotted line) for a 6·1 % aqueous solution of polyvinyl alcohol (Davenport, Richardson & Bradshaw 1967). For $V^{\frac{1}{3}} > 2$ cm interactions with the walls become important.

V is given and a is determined by (10) once U is known. By comparing (10) and U with experiments we effectively compare the shape of the spherical cap as predicted by the theory with the experimental shape. Indeed it is well known that (10) holds accurately (see, for instance, Davenport *et al.* (1967) results for polyvinyl alcohol).

To the zeroth order we obtain from (10) and (13),

$$3V/4\pi a^3 = 20R_e^{-1}, \tag{17}$$

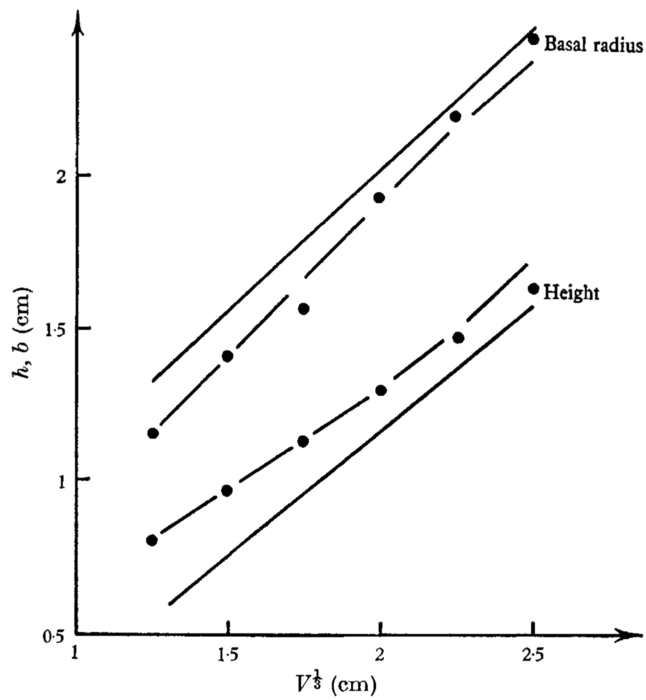

FIGURE 4. Comparison of the height and basal radius of the cap, using (13) (solid lines) and (20) (dashed lines) with experimental results (dotted lines) for a 6·1 % aqueous solution of polyvinyl alcohol (Davenport *et al.* 1967).

which shows that for large Reynolds numbers the bubble is much smaller than the volume within (L). Expanding (14) for small θ gives

$$\theta \simeq (\tfrac{4}{3}80)^{\frac{1}{4}} R_e^{-\frac{1}{4}}. \tag{18}$$

Unless R_e is very large (so that $R_e^{-\frac{1}{4}}$ is small), it is clear from (18) that θ is usually not small. In general θ must be computed from the complete equation (14) (and not from an expansion of it valid for small θ). Figure 4 compares h and b computed from (10), (13) to (16) with experimental results; the agreement is fair.

3. Conclusion

The presence of boundary layers around (L) perturbs (11) by terms of order $R_e^{-\frac{1}{2}}$. Notice in that respect that the error introduced by ignoring the volume of the cap is at most of order R_e^{-1} as can be seen from (17). Assuming that the boundary-layer corrections computed by Harper & Moore (1968), can be used in the present case, we obtain,

$$\phi = 30\pi\mu U^2 a[1+(0\cdot 14\ln R_e - 6\cdot 6)R_e^{-\frac{1}{2}}]. \tag{19}$$

From which we deduce at once

$$U = U_0[1+(6\cdot 6 - 0\cdot 14\ln R_{e0})(9R_{e0})^{-\frac{1}{2}}], \tag{20}$$

where U_0 is the velocity given by (13) and $R_{e0} = 2\rho a U_0/\mu$. Figure 2 shows the excellent agreement between the theoretical velocity (equation (20)) and the experimental results. In particular, (20) represents a significant improvement over (13). Figure 3 shows a similar agreement when $V^{\frac{1}{3}} < 2$ cm (when wall effects are negligible). On figure 4 we also plot h and b using (20) instead of (13).

It is of some interest to express the drag coefficient C_D, based on the sphere of radius r whose volume is that of the bubble, in terms of the Reynolds number. By definition

$$C_D = 2gV/U^2\pi r^2, \tag{21}$$

with $V = 4\pi r^3/3$. We find at once

$$C_D = 6\{20R_e^{-1}[1+(0\cdot 14\ln R_e - 6\cdot 6)R_e^{-\frac{1}{2}}]\}^{\frac{1}{3}}. \tag{22}$$

Equation (22) could be used equally well, instead of (20) to compare the theory with experiments. Equation (22) also points out a basic difference between spherical caps with laminar and turbulent wakes; it shows that the drag coefficient decreases with increasing Reynolds number, while experiments show that the drag coefficient is constant in the turbulent régime (see, for instance, Haberman & Morton 1956). Finally, it should be pointed out that despite the excellent agreement with experiments, there is some uncertainty in applying Harper & Moore's higher order correction to our problem: the logarithmic term may be affected by the presence of the cap, and also, the deformation of (L) from a spherical shape may introduce an additional correction.

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